TURBULENCE UNDER A MAGNIFYING GLASS[1]

Krzysztof Gawędzki

C.N.R.S., I.H.E.S.,
Bures-sur-Yvette, 91440, France

INTRODUCTION

This is an introductory course on the open problems of fully developed turbulence which present a long standing challenge for theoretical and mathematical physics. The plan of the course is as follows:

Lecture 1. Hydrodynamical equations. Existence of solutions. Statistical description. Kolmogorov scaling theory.

Lecture 2. Functional approach to turbulence, similarities and differences with field theory.

Lecture 3. Passive scalar and breakdown of the Kolmogorov theory.

Lecture 4. Inverse renormalization group.

LECTURE 1

The hydrodynamical flows in gasses and liquids are believed to be described in a variety of realistic situations by incompressible Navier-Stokes (NS) equations

$$\partial_t v + (v\cdot\nabla)v - \nu\Delta v = \tfrac{1}{\rho}(f - \nabla p)\,, \qquad \nabla\cdot v = 0 \tag{1}$$

where the vector $v(t,x)$ describes the velocity of the fluid at time t and space point x, positive constant ν is the viscosity, ρ is the constant density[2], $f(t,x)$ is the (intensive) force and $p(t,x)$ is the pressure. The above equations date back to the works of Navier (1823) and Stokes (1843) and modify the even older Euler equation (1755) by the addition of the dissipative term $\nu\Delta v$. In most physical applications the dimensionality of the space is 3 or 2, but the equations make sense in general dimension d. The Euler equation without the $\nu\Delta v$ term is really the $F = ma$ (or rather $a = \frac{1}{m}F$) equality for the volume element of the fluid. The $\nu\Delta v$ term represents the friction forces. By taking the L^2 scalar product of the NS equation with v and assuming that the flow velocity vanishes sufficiently fast at large distances (or at the boundary), one deduces the energy balance:

$$\partial_t \tfrac{1}{2}\int v^2 = -\nu\int(\nabla v)^2 + \int f\cdot v\,. \tag{2}$$

[1]lectures given at the 1996 Cargèse Summer Institute, July 22 - August 3

[2]we shall set it to 1 below

The equation says that the rate of change of fluid energy is equal to the energy injection $\int f \cdot v$ (the work of external forces per unit time) minus the energy dissipation per unit time $\nu \int (\nabla v)^2$ due to the fluid friction.

The Euler equation has a nice infinite-dimensional geometric interpretation: it describes the geodesic flow on the group of volume preserving diffeomorphisms[3]. We shall briefly sketch this argument. Let, more generally, the space be an oriented d-dimensional Riemannian manifold (M, g). Denote by τ the Riemannian volume: $\tau(x) = \sqrt{g(x)}\, dx^1 \wedge \cdots \wedge dx^d$. Let $Diff_\tau$ be the group of diffeomorphisms ϕ of M preserving τ: $\phi^*\tau = \tau$. The space $Vect_\tau$ of divergenceless vector fields v (i.e. the ones preserving τ, $\mathcal{L}_v \tau = 0$) may be considered as the Lie algebra of $Diff_\tau$[4]. Upon the identification of infinitesimal variations of diffeomorphisms ϕ with the vector fields v by $\delta\phi(x) = v(\phi(x))$, the scalar product of vector fields

$$\|v\|^2 = \int\limits_M g(v, v)\, \tau \equiv \int\limits_M g_{ij}(x)\, v^i(x)\, v^j(x)\, \tau(x)$$

induces on $Diff_\tau$ a right-invariant Riemannian metric. This metric in not left-invariant. Indeed,

$$dL_\phi\, dR_{\phi^{-1}}\, v = \phi_* v$$

where dL_ϕ and dR_ϕ stand for the tangent maps to the left and right translations by ϕ on $Diff_\tau$ and

$$\phi_* v(x) = d\phi\, v(\phi^{-1}(x))$$

is the pushforward of v by ϕ. But, in general,

$$\|\phi_* v\|^2 \neq \|v\|^2$$

since the tangent map $d\phi$ does not preserve the Riemannian length of vectors.

The geodesic flows with respect to left-right-invariant metrics on a group are, modulo time-independent left and right translations, one-parameter subgroups. This is not the case if the metric is only right-invariant. The geodesic motions $t \mapsto \phi_t$ on $Diff_\tau$ corresponding to the right-invariant Riemannian metric defined above extremize the action

$$S(\phi) = \tfrac{1}{2} \int \|v(t)\|^2\, dt$$

where

$$\partial_t \phi(t, x) = v(t, \phi(t, x)) \equiv v(t, y)\,.$$

Explicitly[5],

$$\delta S(\phi) = \tfrac{1}{2}\, \delta \int g_{ij}(y)\, v^i(t, y)\, v^j(t, y)\, \tau(y)\, dt$$

[3]recall that the Euler top describes the geodesic flow on the group $SO(3)$

[4]for non-compact M we shall assume that ϕ do not move points outside a compact subset and that v have compact support

[5]field theoriests will notice a similarity of the following calculation to those in nonlinear sigma models

$$
\begin{aligned}
&= \tfrac{1}{2}\,\delta \int g_{ij}(\phi(t,x))\,\partial_t\phi^i(t,x)\,\partial_t\phi^j(t,x)\,\tau(x)\,dt \\
&= \tfrac{1}{2}\int \partial_k g_{ij}(\phi(t,x))\,\delta\phi^k(t,x)\,\partial_t\phi^i(t,x)\,\partial_t\phi^j(t,x)\,\tau(x)\,dt \\
&\qquad\qquad + \int g_{ij}(\phi(t,x))\,\partial_t\phi^i(t,x)\,\partial_t\delta\phi^j(t,x)\,\tau(x)\,dt \\
&= \int \delta\phi^j(t,x)\Big[\tfrac{1}{2}\,\partial_j g_{ik}(y)\,v^i(t,y)\,v^k(t,y) \,-\, \partial_k g_{ij}(y)\,v^i(t,y)\,v^k(t,y) \\
&\qquad\qquad - g_{ij}(y)\,\Big(\partial_t v^i(t,y) + v^k(t,y)\partial_k v^i(t,y)\Big)\Big]\tau(y)\,dt \\
&= -\int g_{ij}\,u^j\Big[\partial_t v^i \,+\, v^k\partial_k v^i \,+\, \tfrac{1}{2}g^{in}(\partial_k g_{ln} + \partial_l g_{kn} - \partial_n g_{lk})\,v^k\,v^l\Big]\tau\,dt \\
&= -\int g(\,u\,,\,\partial_t + \nabla_v v\,)\,\tau\,dt
\end{aligned}
$$

where $u^j(t,y) \equiv \delta\phi^j(t,x)$ is an arbitrary divergenceless vector field.

$$
(\nabla_w v)^i = w^k\partial_k v^i + \{{}^{\,i}_{k\,l}\}\,w^k v^l
$$

denotes the covariant derivative of the vector field v in the direction of w with the Levi-Civita symbols

$$
\{{}^{\,i}_{k\,l}\} \,=\, \tfrac{1}{2}g^{in}(\partial_k g_{ln} + \partial_l g_{kn} - \partial_n g_{lk})\,.
$$

Since a vector field on M orthogonal to all vector fields without divergence is a gradient, we obtain the (generalized) Euler equation

$$
\partial_t v + \nabla_v v = -\nabla p
$$

for the divergenceless vector fields v. The NS term $\nu\Delta v$ still makes sense in the geometric setup and may be added to the equation.

The Euler and NS equations are examples of nonlinear partial differential evolution equations. After a century and a half study, they still pose major open problems as far as the control of their solutions is concerned. The most interesting questions concern the short-distance (ultra-violet) behavior. Suppose that we start from smooth initial data and the force f is smooth. For simplicity, let us assume compact support of both (we may also consider the compact space or the case with boundary conditions). It is known that the smooth solutions of the so posed problems are unique and exist for short time. Do they exist for all times? The answer is positive in 2 dimensions for both Euler and NS equations [43, 20] but in 3 dimensions the answer is not known. It is expected to be positive in the NS case. The opinions about the Euler case (no blowup versus finite-time blowup for special smooth initial conditions) are divided and fluctuate in time.

In an important 1933 paper on the NS equation Leray [27] has introduced the notion of weak solutions of the equation. A vector field $v(t,x)$ locally in L^2 is a weak solution if

$$
\int \Big[\Big(\partial_t u^i + \nu\Delta u^i + (\partial_j u^i)v^j\Big)\,v^i \,+\, u^i f^i\Big] \,=\, 0 \qquad\text{and}\qquad \int (\partial_i\varphi)v^i \,=\, 0
$$

for any smooth divergenceless vector field u and any smooth function φ, both with compact supports. Leray showed by compactness arguments existence of weak global solutions of the 3-dimensional NS equations with additional properties (e.g. with space

derivatives locally square integrable). The weak solutions are not unique (there are weak solutions of the 2-dimensional Euler equation with compact support [36, 37]).

The NS equation is invariant under rescalings. Let

$$\begin{aligned}\widetilde{v}(t,x) &= \tau s^{-1}\, v(\tau t, sx)\,,\\ \widetilde{f}(t,x) &= \tau^2 s^{-1}\, f(\tau t, sx)\,,\\ \widetilde{p}(t,x) &= \tau^2 s^{-2}\, v(\tau t, sx)\,,\\ \widetilde{\nu} &= \tau s^{-2}\,\nu\,.\end{aligned}$$

If v and p solve the NS equation with viscosity ν and force f then $\widetilde{v}$ and $\widetilde{p}$ give a solution for viscosity $\widetilde{\nu}$ and force $\widetilde{f}$. It is convenient to introduce the dimensionless version of the (inverse) viscosity, the Reynolds number

$$R = \frac{\delta_{_L} v\, L}{\nu}\,,$$

where $\delta_{_L} v$ is a characteristic velocity difference over scale L of the order of the size of the system. For example, for the flow in a pipe of radius L, we may take $\delta_{_L} v$ as velocity in the middle of the pipe (minus the vanishing velocity on the wall of the pipe). A basic phenomenological observation in hydrodynamics is that for $R \lesssim 1$ the flows are regular (laminar) whereas $R \gg 1$ correspond to very irregular (turbulent) flows with a rich set of scenarios occurring for intermediate R. Note the scale-dependent character of the Reynold number. Following [17], define the running Reynolds number

$$R_r = \left(\frac{1}{|\mathcal{O}(r)|}\int_{\mathcal{O}(r)} |\nabla v|^2\right)^{1/2} \frac{r^2}{\nu}$$

where $\mathcal{O}(r) = \{\, (s,y) \mid |s-t| < \frac{r^2}{\nu},\ |y-x| < r\,\}$. Note that we may rewrite

$$R_r = \frac{\delta_r v\ r}{\nu}$$

where $\delta_r v$, the mean velocity difference on scale r, is calculated by multiplying the mean square gradient of v over $\mathcal{O}(r)$ by r. The best regularity result about the weak solutions of the NS equation is due to Caffarelli-Kohn-Nirenberg [6] and says that there exist $\epsilon > 0$ such that if $R_r < \epsilon$ then the solution is smooth on the $\mathcal{O}(\epsilon r)$ neighborhood of (t,x) [17]. This implies that for a weak solution, the Haussdorf dimension of the set of singularities is < 1. Note the spirit of the result in line with the phenomenological characterization of laminar flows.

For high Reynolds numbers it is reasonable to attempt a statistical description of complicated turbulent flows. In the theoretical approach, the statistics may be generated by considering random initial data or/and random forcing. $v(t,x)$ becomes then a random field. We shall be interested in describing a stationary statistical state of the latter. In such a state the mean overall energy of the fluid is constant in time so that the energy balance equation (2) implies that

$$\int \langle \nu (\nabla v)^2 \rangle = \int \langle f \cdot v \rangle\,,$$

where $\langle\, - \,\rangle$ denotes the ensemble average, or, in a homogeneous state,

$$\bar{\epsilon} \equiv \langle \nu (\nabla v)^2 \rangle = \langle f \cdot v \rangle \equiv \bar{\varphi} \tag{3}$$

where $\bar{\epsilon}$ denotes the mean dissipation rate and $\bar{\varphi}$ the mean injection rate of energy, both with dimension $\frac{length^2}{time^3}$. In the situation where the energy injection is a large distance process (e.g. in the atmospheric turbulence) one expects that for high R a scale separation occurs with the energy dissipation taking place on much smaller distances. Pictorially, energy is transmitted to the fluid by induction of large eddies on scale L which subsequently break to smaller scale eddies and so on passing energy to shorter and shorter scales without substantial loss until the viscous scale η is reached where the friction dissipates energy. Such a (Richardson [34]) energy cascade is then characterized by the integral scale L, the viscous scale η and the energy dissipation rate $\bar{\epsilon}$. The scale ratio L/η should grow with the Reynolds number. The interval of distance scales r satisfying $L \gg r \gg \eta$ is called the inertial range. The cascade picture may be formulated in more quantitative terms by introducing the quantities [16]

$$\bar{\epsilon}_{\leq K} \;=\; \nu \int\limits_{|k|\leq K} \Big(\int \langle \nabla v(0)\cdot\nabla v(x)\rangle \, \mathrm{e}^{-i\,k\cdot x}\, dx\Big) đk$$

($đk \equiv \frac{dk}{(2\pi)^{-d}}$) and

$$\bar{\varphi}_{\leq K} \;=\; \int\limits_{|k|\leq K} \Big(\int \langle f(0)\cdot v(x)\rangle \, \mathrm{e}^{-i\,k\cdot x}\, dx\Big) đk$$

interpreted as the mean dissipation and mean injection rate in wavenumbers k with $|k| \leq K$. The injection of energy limited to distances $\gtrsim L$ means that, as a function of K, $\bar{\varphi}_{\leq K}$ is close to $\bar{\epsilon}$ everywhere except for $K \lesssim \frac{1}{L}$ where it falls to zero with $K \to 0$. Similarly, the cascade picture should imply that $\bar{\epsilon}_{\leq K}$ is negligable for $K \ll \frac{1}{\eta}$ and then grows to $\bar{\epsilon}$. The difference

$$\bar{\varphi}_{\leq K} - \bar{\epsilon}_{\leq K} \;\equiv\; \bar{\pi}_K$$

has the interpretation of the energy flux out of the wavenumbers k with $|k| \leq K$. It should be approximately constant and equal to $\bar{\epsilon}$ in the inertial range $\frac{1}{L} \ll K \ll \frac{1}{\eta}$.

In 1941, Kolmogorov [23] went one step further by postulating that the equal-time correlators of velocity differences over distances in the inertial range should be universal functions of the latter and of the dissipation rate $\bar{\epsilon}$. In particular for the structure functions

$$S_n(x) \;=\; \langle ((v(x) - v(0))\cdot\widehat{x})^n\rangle$$

with $\widehat{x} \equiv \frac{x}{|x|}$, assuming also isotropy of the turbulent state, one obtains

$$S_n(x) \;=\; C_n\, \bar{\epsilon}^{n/3} r^{n/3} \tag{4}$$

with $r \equiv |x|$ and universal constants C_n. Indeed, the right hand side is the only function of $\bar{\epsilon}$ and r with the dimension $(\frac{length}{time})^n$. Kolmogorov theory implies that the typical velocity difference over distance r behaves as $\bar{\epsilon}^{1/3} r^{1/3}$. For the scale-dependent Reynolds number one obtains then $R_r \sim \frac{\bar{\epsilon}^{1/3} r^{4/3}}{\nu}$. In particular, $R \cong R_L \sim \frac{\bar{\epsilon}^{1/3} L^{4/3}}{\nu}$ and $\mathcal{O}(1) = R_\eta \sim \frac{\bar{\epsilon}^{1/3}\eta^{4/3}}{\nu}$ hence $\frac{\eta}{L} \sim R^{-3/4}$ and it decreases with R.

For $n = 3$ the relation (4) is essentially a rigorous result with $C_3 = -\frac{4}{5}$ in 3 dimensions. Let us sketch the latter argument [26, 16]. Suppose for simplicity that the force f is a random Gaussian field with mean zero and the covariance

$$\langle f^i(t,x)\, f^j(s,y)\rangle \;=\; \delta(t-s)\, \mathcal{C}^{ij}(x-y) \tag{5}$$

with $\partial_i \mathcal{C}^{ij} = 0$. Inferring from the NS equation that

$$v(t+\Delta t) \,=\, v(t) \,-\, ((v\cdot\nabla v)v - \nu\Delta v + \nabla p)|_t \Delta t \,+\, \int_t^{t+\Delta t} f(s)\, ds \,+\, \mathcal{O}(\Delta t)^2$$

the stationarity of $\langle v(t,x)\cdot v(t,y)\rangle$ implies that

$$2\nu\,\langle \nabla v(x)\cdot\nabla v(y)\rangle \,=\, -\nu\langle \Delta v(x)\cdot v(y)\rangle \,-\, \nu\langle v(x)\cdot \Delta v(y)\rangle$$

$$=\, -\langle (v(x)\cdot\nabla)v(x)\cdot v(y)\rangle \,-\, \langle v(x)\cdot(v(y)\cdot\nabla)v(y)\rangle \,+\, \mathrm{tr}\,\mathcal{C}(x-y)$$

(the pressure does not contribute due to the transversality of v). The first two terms on the right hand side may be rewritten as

$$\tfrac{1}{2}\nabla_x\cdot\langle (v(x)-v(y))^2(v(x)-v(y))\rangle$$

so that we obtain the relation

$$\nu\,\langle \nabla v(x)\cdot\nabla v(0)\rangle \,-\, \tfrac{1}{4}\nabla_x\cdot\langle (v(x)-v(0))^2(v(x)-v(0))\rangle \,=\, \tfrac{1}{2}\,\mathrm{tr}\,\mathcal{C}(x)\,. \tag{6}$$

Taking first the limit $x\longrightarrow 0$ for positive ν and assuming that the presence of the latter smoothes the behavior of $\langle (v(x)-v(0))^2(v(x)-v(0))\rangle$ so that the second term on the left hand side vanishes when $x\longrightarrow 0$, we obtain

$$\bar\epsilon \,=\, \tfrac{1}{2}\,\mathrm{tr}\,\mathcal{C}(0)$$

which is nothing else then the energy balance equation (3). On the other hand, taking first the $\nu\longrightarrow 0$ limit of eq. (6) for $x\neq 0$, we obtain

$$-\,\tfrac{1}{4}\nabla_x\cdot\langle (v(x)-v(0))^2(v(x)-v(0))\rangle|_{\nu=0} \,=\, \tfrac{1}{2}\,\mathrm{tr}\,\mathcal{C}(x)\,. \tag{7}$$

The assumption that the force acts only on distances $\gtrsim L$ means that for $r\ll L$, $\mathcal{C}(x)\cong\mathcal{C}(0)$ so that eq. (7) means that in the inertial range

$$-\,\tfrac{1}{4}\nabla_x\cdot\langle (v(x)-v(0))^2(v(x)-v(0))\rangle \,=\, \bar\epsilon\,.$$

Together with isotropy, this implies the relation [16]

$$\langle (v^i(x)-v^i(0))(v^j(x)-v^j(0))(v^k(x)-v^k(0))\rangle = -\tfrac{4\,\bar\epsilon}{d(d+2)}\,(\delta^{ij}x^k+\delta^{ik}x^j+\delta^{jk}x^i)$$

from which the $n=3$ case of eq. (4) follows with $C_3 = -\frac{12}{d(d+2)}$.

The structure functions are measured, more or less directly, in atmospheric or ocean flows, in water jets, in aerodynamic tunnels or in subtle experiments with helium gas inbetween rotating cylinders. They are also accessible in numerical simulations. One extracts then the scaling exponents assuming the behavior

$$S_n(x)\,\sim\, r^{\zeta_n}\,.$$

ζ_3 agrees well with the theoretical prediction $\zeta_3 = 1$. Here are some other exponents obtained from wind tunnel data [3]

$$\zeta_2 = .70\,(.67),\quad \zeta_4 = 1.28\,(1.33),\quad \zeta_5 = 1.53\,(1.67),\quad \zeta_6 = 1.77\,(2),\quad \zeta_7 = 2.01\,(2.33)$$

with the Kolmogorov values in the parenthesis for comparison. The discrepancy is quite pronounced (its direction is determined by the Hölder inequality). One of the main open problems in the theory of fully developed turbulence is to explain, starting from the first principles (i.e. from the NS equation), the breakdown of the Kolmogorov theory leading to the anomalous structure-function exponents which indicate that the distribution of $v(t)$ in the inertial range is far from Gaussian. The intuitive idea that only a part of fluid modes (temporal or/and spacial) participates in the turbulent cascade ("intermittency") has led to multiple models of the cascade of the (multi)fractal type [16]. Such models are not based on the NS equation and allow to obtain essentially arbitrary spectra of (multifractal [12]) exponents hence they do not really explain the breakdown of the normal scaling in realistic flows.

For the "energy spectrum"

$$e(K) \equiv \tfrac{1}{2}\tfrac{d}{dK}\int\limits_{|k|\leq K}\Big(\int \langle v(0)\cdot v(x)\rangle\, \mathrm{e}^{-i\,k\cdot x}\, dx\Big) đk\,,$$

the Kolmogorov theory predicts

$$e(K) \sim \bar{\epsilon}^{2/3}\, K^{-5/3}$$

for $\frac{1}{L} \ll K \ll \frac{1}{\eta}$. The experimental data seem to confirm this behavior (with the possibility of a slight discrepancy consistent with the above value of ζ_2). Deep in the dissipative regime ($K \gg \frac{1}{\eta}$), $e(K)$ falls off much faster than in the inertial regime.

LECTURE 2

The NS equation with random force is an example of a stochastic evolution equation of the form

$$\partial_t\Phi \,=\, G(\Phi) \,+\, f \tag{8}$$

with Φ a function of time, $G(\Phi)$ a functional local in time and f a stationary Gaussian process with mean zero and covariance

$$\langle\, f(t)\, f(s)\,\rangle \,=\, C(t-s)\,.$$

Upon solving the equation for given f with initial data $\Phi(t_0) = \Phi_0$, one obtains a stochastic process $t \mapsto \Phi(t)$. The limit $t_0 \to -\infty$, if exists, should then allow to pass to the stationary regime without dependence on the initial data. There is a simple way to write the expectation values of a functional $\mathcal{F}$ of Φ in terms of a formal functional integral:

$$\langle\, \mathcal{F}(\Phi)\,\rangle \,=\, \int \mathcal{F}(\Phi)\; \delta(\partial_t\Phi - G(\Phi) - f)\; \det(\partial_t - \tfrac{\delta G}{\delta\Phi})\; D\Phi\; d\mu_{_C}(f) \tag{9}$$

where $d\mu_{_C}$ denotes the Gaussian measure with covariance C. Indeed, the integration over Φ imposes the dependence (8) between Φ and f and the f integral calculates the Gaussian expectation of $\mathcal{F}(\Phi)$ viewed as a functional of f. The determinant in (9) is calculable (formally).

$$\det(\partial_t - \tfrac{\delta G}{\delta\Phi}) \,=\, \det(\partial_t)\; \det(1 - \partial_t^{-1}\tfrac{\delta G}{\delta\Phi})$$

$$= \det(\partial_t)\, \mathrm{e}^{\mathrm{tr}\, \ln(1-\partial_t^{-1}\frac{\delta G}{\delta \Phi})} = \det(\partial_t)\, \mathrm{e}^{-\sum\limits_{n=1}^{\infty} \mathrm{tr}\,(\partial_t^{-1}\frac{\delta G}{\delta\Phi})^n} .$$

With the choice $\partial_t^{-1}(t_1, t_2) = \theta(t_1 - t_2)$, appropriate for solutions of the initial value problems, we have, setting[6] $\theta(0) = \frac{1}{2}$,

$$\mathrm{tr}\,(\partial_t^{-1}\tfrac{\delta G}{\delta\Phi})^n \;=\; \begin{cases} 0 & \text{for} \quad n > 1\,, \\ \frac{1}{2}\int \mathrm{tr}\,\frac{\delta G}{\delta\Phi}\, dt & \text{for} \quad n = 1\,, \end{cases}$$

where on the right hand side, $\frac{\delta G}{\delta\Phi}$ is calculated at fixed time. Inserting the value of the determinant into eq. (9) and rewriting the δ-function as an oscillatory integral, we obtain

$$\langle\, \mathcal{F}(\Phi)\,\rangle \;=\; \int \mathcal{F}(\Phi)\; \mathrm{e}^{-i\,(R,\, \partial_t\Phi - G(\Phi) - f\,) - \frac{1}{2}\int \mathrm{tr}\,\frac{\delta G}{\delta\Phi}\, dt}\; DR\, D\Phi\; d\mu_{_C}(f) \;\Big/\; \text{norm.},$$

where $(\cdot\,,\,\cdot)$ stands for the L^2 scalar product, or, after integration over f,

$$\langle\, \mathcal{F}(\Phi)\,\rangle \;=\; \int \mathcal{F}(\Phi)\; \mathrm{e}^{-S(R,\Phi)}\; DR\, D\Phi \;\Big/\; \text{norm.} \tag{10}$$

where

$$S(R,\Phi) \;=\; i\,(\,R,\, \partial_t\Phi - G(\Phi)\,) \,+\, \tfrac{1}{2}\int \mathrm{tr}\,\tfrac{\delta G}{\delta\Phi}\, dt \,+\, \tfrac{1}{2}\,(R, CR)\,. \tag{11}$$

The above functional integral representation is known in the physical literature as the Martin-Siggia-Rose (MSR) formalism [30]. The field R is called the response field (its correlations measure the reaction of the system to small deterministic variations of the force). It may be integrated out in (10) leading to a Fokker-Planck type of functional integral but it will be more convenient to keep it in the functional representation.

As a simple but instructive example, let us consider a linear stochastic equation describing forced diffusion:

$$\partial_t T \;=\; D\Delta T \,+\, f \tag{12}$$

with the diffusion constant D and the forcing covariance

$$\langle\, f(t,x)\, f(s,y)\,\rangle \;=\; \delta(t-s)\,\mathcal{C}(x-y) \tag{13}$$

where $\mathcal{C}$ is a smooth, fast decaying, positive-definite function. Given the initial data $T(t_0)$, we may solve eq. (12):

$$T(t,x) \;=\; \int \mathrm{e}^{(t-t_0)D\Delta}(x,y)\; T(t_0,y)\; dy \,+\, \int\limits_{t_0}^{t} ds \int \mathrm{e}^{(t-s)D\Delta}(x,y)\; f(s,y)\; dy$$

obtaining a Gaussian stochastic process with mean

$$\mathrm{e}^{(t-t_0)D\Delta}\, T(t_0)$$

and covariance

$$\langle\, (T(t_1,x_1) - \langle T(t_1,x_1)\rangle)\, (T(t_2,x_2) - \langle T(t_2,x_2)\rangle)\,\rangle$$

[6] in simplest situations, this choice cancels in final expressions, in more complicated ones it is related to a choice of the (ordering) convention for the stochastic integrals

$$= \int_{t_0}^{\min(t_1,t_2)} \left(e^{(t_1-s)D\Delta}\,\mathcal{C}\,e^{(t_2-s)D\Delta}\right)(x_1,x_2)\,ds\,.$$

The limit $t_0 \to -\infty$ results in a stationary Gaussian process with mean zero and covariance

$$\langle\, T(t_1,x_1)\,T(t_2,x_2)\,\rangle \;=\; \int e^{|t_1-t_2|\,Dk^2\,-\,ik\cdot(x_1-x_2)}\;\frac{\widehat{\mathcal{C}}(k)}{2Dk^2}\,đk\,. \tag{14}$$

Note that for $t_1 = t_2$ the right hand side is a propagator of the scalar massless free field with UV cutoff $\widehat{\mathcal{C}}(k)$.

In the MSR formalism, the stochastic equation (12) leads to the quadratic action

$$S(R,T) = i\!\int\! R(t,x)\,(\partial_t T - D\Delta T)(t,x)\,dt\,dx + \tfrac{1}{2}\!\int\! R(t,x)\,\mathcal{C}(x-y)\,R(t,y)\,dt\,dx\,dy$$

which gives rise to the propagators

$$\begin{aligned}
\langle\, R\,R\,\rangle \;&=\; 0\,,\\
\langle\, T(t_1,x_1)\;R(t_2,x_2)\,\rangle \;&=\; i(\partial_t + D\Delta)^{-1}(t_1,x_1;t_2,x_2)\\
&=\; i\theta(t_1-t_2)\;e^{(t_1-t_2)D\Delta}(x_1,x_2)
\end{aligned} \tag{15}$$

and $\langle TT\rangle$ as in eq. (14).

There are two interesting limiting regimes of forced diffusion: the one of long times and long distances and that of short times and short distances. To study the first one, let us introduce the rescaled fields

$$T_\lambda(t,x) \;=\; \lambda^{(d-2)/2}\,T(\lambda t^2,\,\lambda x)\,, \qquad R_\lambda(t,x) \;=\; \lambda^{(d+2)/2}\,R(\lambda^2 t,\,\lambda x) \tag{16}$$

for which

$$\begin{aligned}
&\langle\, T_\lambda(t_1,x_1)\,T_\lambda(t_2,x_2)\,\rangle \;=\; \lambda^{d-2}\int e^{|t_1-t_2|\lambda^2 Dk^2\,-\,ik\cdot(x_1-x_2)\lambda}\;\frac{\widehat{\mathcal{C}}(k)}{2Dk^2}\,đk\\
&= \int e^{|t_1-t_2|\,Dk^2\,-\,ik\cdot(x_1-x_2)}\;\frac{\widehat{\mathcal{C}}(k/\lambda)}{2Dk^2}\,đk \;\;\underset{\lambda\to\infty}{\longrightarrow}\;\; \frac{\widehat{\mathcal{C}}(0)}{2D}\int e^{-|t_1-t_2|\,Dk^2\,+\,ik\cdot(x_1-x_2)}\;\frac{1}{k^2}\,đk\,,
\end{aligned} \tag{17}$$

$$\langle\, T_\lambda\,R_\lambda\,\rangle \;=\; \langle\, T\,R\,\rangle\,.$$

The limiting $\langle T_\lambda T_\lambda\rangle$ covariance coincides with the one of the Langevin dynamics (of type A [21]) for the scalar massless field. Recall that the Langevin dynamics is given by the stochastic evolution equation

$$\partial_t\Phi \;=\; -\tfrac{1}{2}\,\Gamma\,\frac{\delta\mathcal{V}(\Phi)}{\delta\Phi} \;+\; f$$

with the delta-correlated noise

$$\langle\, f(t,x)\,f(s,y)\,\rangle \;=\; \Gamma\,\delta(t-s)\,\delta(x-y)\,.$$

For $\mathcal{V}(\Phi) = \int V(\Phi(t))\,dt$, it describes convergence to the equilibrium (equal time) probability measure $\sim e^{-V(\Phi)}\,D\Phi$. In our case

$$V(T) \;=\; \frac{D}{\widehat{\mathcal{C}}(0)}\int(\nabla T)^2\,, \qquad\qquad \Gamma \;=\; \widehat{\mathcal{C}}(0)\,.$$

The limiting behavior (17) expresses the simple fact that at large distances the noise covariance $\mathcal{C}(x-y)$ looks like $(\int\mathcal{C})\,\delta(x-y)$.

The study of the long-time, long-distance asymptotics is a typical field-theoretic problem where we are interested in the behavior of the system at distances much longer than the UV cutoff scale. Studying the opposite regime of distances much shorter than the cutoff scale may seem without interest. This is not so. In problems related to turbulence, function $\widehat{\mathcal{C}}(k)$ describes the spectrum of forcing rather than the momentum cutoff and we are interested in the regime of short times and short distances, more exactly in distance scales much shorter than the scale of energy injection interpreted in the field-theoretic context as the UV cutoff scale. Hence **field theory and turbulence are concerned with opposite limiting regimes**. In order to examine the short-time, short-distance asymptotics of forced diffusion, we rescale the fields differently introducing

$$T^\lambda(t,x)\,=\,\lambda T(t/\lambda^2,x/\lambda)\,,\qquad R^\lambda(t,x)\,=\,\lambda^{-d-1}R(t/\lambda^2,x/\lambda)\,. \tag{18}$$

and obtaining

$$\begin{aligned}\langle T^\lambda(t_1,x_1)\,T^\lambda(t_2,x_2)\rangle\,&=\,\lambda^2\int \mathrm{e}^{|t_1-t_2|\lambda^{-2}Dk^2\,-\,ik\cdot(x_1-x_2)\lambda^{-1}}\,\frac{\widehat{\mathcal{C}}(k)}{2Dk^2}\,đk\\ &=\,\lambda^2\int\frac{\widehat{\mathcal{C}}(k)}{2Dk^2}\,đk\,-\,\tfrac12\mathcal{C}(0)\left(|t_1-t_2|\,+\,\tfrac{1}{Dd}\,|x_1-x_2|^2\right)\,+\,\mathcal{O}(\lambda^2)\,,\\ \langle T^\lambda\,R^\lambda\rangle\,&=\,\langle T\,R\rangle\end{aligned}\tag{19}$$

for large λ. Hence the $\langle T^\lambda T^\lambda\rangle$ covariance reaches a scaling form modulo a divergent constant. In other words, the differences $T(t,x)-T(t,y)$ exhibit a scaling behavior at short times and short distances. The presence of the blowing up constant mode which may be eliminated by considering field differences is typical for the turbulence related problems.

Returning to the NS equation (1) with random Gaussian force (5) and applying the MSR formalism, we obtain the MSR action

$$\begin{aligned}S(R,v)\,=\,i\int R(t,x)\cdot(\partial_t v+v\cdot\nabla v-\nu\Delta v)(t,x)\,dt\,dx\\ +\tfrac12\int R(t,x)\cdot\mathcal{C}(x-y)\,R(t,y)\,dt\,dx\,dy\end{aligned}$$

with vector fields R and v satisfying $\nabla\cdot R=0$ and $\nabla\cdot v=0$. One could then set up a perturbative scheme by separating

$$S(R,v)\,=\,S_0(R,v)\,+\,S_1(R,v)$$

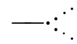
with

$$\begin{aligned}S_0(R,v)&=i\!\int\! R(t,x)\cdot(\partial_t v-\nu\Delta v)(t,x)\,dt\,dx+\tfrac12\!\int\! R(t,x)\cdot\mathcal{C}(x-y)\,R(t,y)\,dt\,dx\,dy,\\ S_1(R,v)&=i\!\int\! R\,P_{\mathrm{tr}}(v\cdot\nabla v)\,dt\,dx,\end{aligned}$$

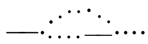
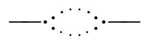
(P_{tr} stands for the projection on the transverse vector fields) and developing e^{-S_1} in the power series, with S_0 giving rise to the propagators

$$\langle R\,v\rangle_0\qquad \text{——}\cdots\quad,\qquad\qquad \langle v\,v\rangle_0\qquad \cdots\cdots$$

and S_1 to the vertex



and with the simplest Feynman diagrams

 ,  .

The perturbative expansion is plagued by the UV and IR divergences when $\nu \to 0$ and $\mathcal{C} \to$ const. Attempts were made to improve the situation by applying various resummation schemes, inspired by the field theoretic techniques (Schwinger-Dyson equations, renormalization group, etc.) but it seems fair to say that they were not very successful.

Despite a formal similarity with the field theory formulation using functional integrals, the problem of the NS turbulence differs radically from the (dynamical formulation) of field theory in at least two crucial aspects:

I. the nonlinear term $v \cdot \nabla v$ in the NS equation is not of the gradient type $\frac{\delta \mathcal{V}(v)}{\delta v}$ unlike the nonlinearities in the Langevin dynamics for field theory models,

II. the noise (force) correlation in the NS turbulence should be close to a delta-function in the Fourier space rather than in the position space as for the Langevin dynamics.

Consequently, the stationary statistical state of the NS fluid is not of an equilibrium type unlike the Gibbs states corresponding to the stationary states of the Langevin dynamics for (euclidean) field theories. We have seen on the example of linear forced diffusion that the 2^{nd} difference is already enough to make the problem very different from the field theoretic one. Let us profit from the occasion to mention that the study of the nonlinear version of forced diffusion:

$$\partial_t T \,=\, D\Delta T \,-\, \frac{1}{2\widehat{\mathcal{C}}(0)}\,\frac{\delta \mathcal{V}(T)}{\delta T} \,+\, f \tag{20}$$

with random Gaussian force as in (13) is a problem of its own interest. The long-time long-distance asymptotics of the correlations is then a dynamical version of an interacting field theory problem (e.g. for $\mathcal{V}(T) \sim \int T^4$) and may be studied by the renormalization group techniques. On the other hand, the short-time, short-distance asymptotics e.g. for $\mathcal{V}(T) \sim \int (\nabla T)^4$ is a completely open problem[7]. The inverse renormalization group which will be the subject of the last lecture may provide a tool for the latter type of questions.

LECTURE 3

In view of the reputed difficulty of the NS problem, it would be instructive to consider simpler models of the stochastic evolution equation (8) randomly forced at long distances and with $G(\Phi)$ not of the gradient type. Such a model is provided by an equation, with a relatively long history [33, 1, 10], describing the passive advection of a scalar quantity $T(t,x)$ (temperature) by a random velocity field

$$\partial_t T \,=\, -v \cdot \nabla T \,+\, \kappa \Delta T \,+\, f\,. \tag{21}$$

[7] we consider functionals of ∇T to avoid coupling to the constant mode unstable at short distances already in the linear case

The positive coefficient κ is called the molecular diffusivity. For a divergence-free v, operator $-v\cdot\nabla$ is skew-symmetric (conserving the energy $\frac{1}{2}\int T^2$) and the $-v\cdot\nabla T$ term is not of the gradient type, unlike the $\kappa\Delta T$ one corresponding to a negative (energy dissipating) symmetric operator. Ideally, the velocity field $v(t,x)$ should describe a turbulent NS flow but, to simplify radically the problem, one replaces it by a Gaussian random field. In [24] Kraichnan noticed that in the case when v is decorrelated in time the problem becomes exactly soluble in the sense that one may write a closed system of differential equations for the correlation functions of T. In recent years Kraichnan's model of passive scalar has attracted a lot of attention, see e.g. [25, 14, 8, 38, 19], as a prototype of a turbulent system in which one may study analytically the breakdown of a Kolmogorov-type scaling.

We shall describe some results of the approach developed in [18, 19, 5]. The Gaussian velocity field v will be taken of mean zero and covariance

$$\langle\, v^i(t,x)\, v^j(s,y)\,\rangle \;=\; \delta(s-t)\, D^{ij}(x-y) \tag{22}$$

with $\partial_i D^{ij}(x)=0$ and

$$D^{ij}(x) \;=\; 2D_0\,\delta^{ij} \;-\; 2d^{ij}(x) \tag{23}$$

where $d^{ij}(x)\;\sim\; r^\xi$ for small $r\equiv|x|$, $0<\xi<2$. Note the implied growth of the $2^{\rm nd}$ order velocity structure function with the distance, as in the turbulent flows. $\xi=\frac{4}{3}$ corresponds to the Kolmogorov scaling dimension[8] of v equal to $\frac{1}{3}$. More concretely, we may pose

$$D^{ij}(x)\;\sim\;\int\frac{{\rm e}^{ik\cdot x}}{(k^2+m^2)^{(d+\xi)/2}}\,\left(\delta^{ij}-\tfrac{k^ik^j}{k^2}\right) \bar{d}k\,, \tag{24}$$

with a small IR regulator m, which leads to $D_0\sim m^{-\xi}$ and

$$d^{ij}(x)\equiv d^{ij}_m(x)\;\cong\;\tfrac{D}{d-1}\left((d-1+\xi)\delta^{ij}-\xi\tfrac{x^ix^j}{r^2}\right)r^\xi\equiv d^{ij}_0(x) \tag{25}$$

for small r and some constant D. Unlike D_0 which diverges when $m\longrightarrow 0$, $d^{ij}_m(x)$ possesses the $m\longrightarrow 0$ limit equal to $d^{ij}_0(x)$ and scaling exactly with power ξ.

The source f will be assumed independent of v, Gaussian, with mean zero and with covariance

$$\langle\, f(t,x)\, f(s,y)\,\rangle \;=\; \delta(t-s)\,\mathcal{C}(\tfrac{x-y}{L})$$

for some positive definite, rotationally invariant test function $\mathcal{C}$.

The passive scalar model as set up above exhibits scale separation with the energy cascade in the inertial range of distances $L\gg r\gg\eta=\mathcal{O}((\frac{\kappa}{D})^{1/\xi})\,L$. The even structure functions of the scalar[9] $S_{2n}(r)=\langle(T(t,x)-T(t,0))^{2n}\rangle$ show exponential scaling in the inertial range. At least for small ξ,

$$S_{2n}(r)\;\sim\;r^{\zeta_{2n}} \tag{26}$$

with

$$\zeta_{2n}\;=\;n(2-\xi)\;-\;\tfrac{2n(n-1)}{d+2}\,\xi\;+\;\mathcal{O}(\xi^2)\,. \tag{27}$$

[8] as we shall see in the next lecture, the dimension of $\delta(t-s)$ is $\xi-2$
[9] the odd ones vanish

In particular, the scaling of the $4^{\rm th}$ and higher structure functions is anomalous: the exponents ζ_{2n} deviate from the ones of the Kolmogorov(-Corrsin) scaling theory giving $\zeta_{2n}=n\zeta_2$. Similar results were obtained in [8, 7] for large space dimension d.

The above conclusions are based on an analysis of the differential equations satisfied by the equal-time correlation functions of the scalar T. These equations may be obtained in many ways, for example with help of the MSR formalism. Let us first consider the example of the 2-point function. We have

$$\langle T(t,x_1)\,T(t,x_2)\rangle\;=\;\int\Big(\int T(t,x_1)T(t,x_2)\,{\rm e}^{-S(R,T,v)}\,DR\,DT\Big/{\rm norm.}\Big)\,d\mu_{_D}(v)$$

where

$$S(R,T,v)=i\!\int\! R\,(\partial_tT+v\cdot\nabla T-\kappa\Delta T)\,dt\,dx+\tfrac{1}{2}\!\int\! R(t,x)\,\mathcal{C}(\tfrac{x-y}{L})\,R(t,y)\,dt\,dx\,dy$$

and $d\mu_{_D}(v)$ is the Gaussian measure corresponding to the covariance (22). The normalization of the Gaussian R,T functional integral may be argued to be v-independent. Performing this integral, one obtains

$$\begin{aligned}&\langle T(t,x_1)\,T(t,x_2)\rangle\\ &=\int\Big((\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}\mathcal{C}(-\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}\Big)\,(t,x_1;t,x_2)\;d\mu_{_D}(v)\,.\end{aligned}$$

Expanding into the Neuman series

$$(\pm\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}\;=\;\sum\limits_{m=0}^\infty(\pm\partial_t-\kappa\Delta)^{-1}\left(-v\cdot\nabla\,(\pm\partial_t-\kappa\Delta)^{-1}\right)^m, \tag{28}$$

we may easily compute the $d\mu_{_D}(v)$ expectation using the Wick theorem. The resulting sum may be identified[10] with the Neuman series for $\mathcal{M}_2^{-1}\mathcal{C}$ where $\mathcal{M}_2$ is an operator acting on the functions of x_1,x_2,

$$\begin{aligned}\mathcal{M}_2\;&=\;-\kappa(\Delta_{x_1}+\Delta_{x_2})\;-\;\tfrac{1}{2\delta(0)}\Big(\langle(v(t,x_1)\cdot\nabla_{x_1})^2\rangle+\langle(v(t,x_2)\cdot\nabla_{x_2})^2\rangle\Big)\\ &\qquad\qquad\qquad\qquad-\tfrac{1}{\delta(0)}\,\langle v(t,x_1)\cdot\nabla_{x_1}\,v(t,x_2)\cdot\nabla_{x_2}\rangle\\ &=\;-(\kappa+D_0)(\Delta_{x_1}+\Delta_{x_2})\;-\;2(D_0\delta^{ij}-d^{ij}(x_1-x_2))\,\partial_{x_1^i}\partial_{x_2^j}\,.\end{aligned}$$

The Neuman series develops $\mathcal{M}_2^{-1}$ in powers of the v-expectations. The "tadpole" term $\langle(v(t,x_i)\cdot\nabla_{x_i})^2\rangle$ originates from the Wick contraction of 2 velocities in the Neuman series of the same $(\pm\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}$ operator whereas $\langle v(t,x_1)\cdot\nabla_{x_1}v(t,x_2)\cdot\nabla_{x_2}\rangle$ comes from the mixed contractions.

Higher point correlation functions can be treated the same way:

$$\begin{aligned}&\mathcal{F}_{2n}(x_1,\dots,x_{2n})\;\equiv\;\langle T(t,x_1)\cdots T(t,x_{2n})\rangle\\ &=\sum\limits_{\substack{\rm pairings\\ (\{n_j,m_j\})}}\int\prod\limits_{j=1}^n\Big((\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}\mathcal{C}(-\partial_t+v\cdot\nabla-\kappa\Delta)^{-1}\Big)(t,x_{n_j};t,x_{m_j})\;d\mu_{_D}(v)\end{aligned}$$

and using eq. (28) and the Wick theorem, it is easy to obtain the inductive relation

$$\mathcal{F}_{2n}(x_1,\dots,x_{2n})\;=\;\mathcal{M}_{2n}^{-1}\sum\limits_{1\le n_1<n_2\le 2n}\mathcal{C}(\tfrac{x_{n_1}-x_{n_2}}{L})\;\mathcal{F}_{2n-2}(x_1,\mathop{\dots}\limits_{\widehat{n_1}}\mathop{\dots}\limits_{\widehat{n_2}},x_{2n}) \tag{29}$$

[10] the time-decorrelation of the velocities is crucial at this point

(in a slightly abusive notation) with

$$\mathcal{M}_n \;=\; -(\kappa+D_0)\sum_{m=1}^{n}\Delta_{x_m} \;-\; 2\sum_{1\le m_1<m_2\le n}(D_0\delta^{ij}-d^{ij}(x_{m_1}-x_{m_2}))\,\partial_{x^i_{m_1}}\partial_{x^j_{m_2}}\,.$$

For $\kappa>0$, $\mathcal{M}_n$ are positive $2^{\rm nd}$ order elliptic operators. For $\kappa=0$ they become singular elliptic (their principal symbol looses positive definiteness at coinciding points $x_{m_1}=x_{m_2}$). Note that in the action on translationally invariant functions of $x_1,\dots,x_n$,

$$\mathcal{M}_n \;=\; -\kappa\sum_{m=1}^{n}\Delta_{x_m} \;+\; 2\sum_{1\le m_1<m_2\le n} d^{ij}(x_{m_1}-x_{m_2})\,\partial_{x^i_{m_1}}\partial_{x^j_{m_2}}\,. \tag{30}$$

Hence in the translationally invariant sector the constant $D_0\sim m^{-\xi}$ divergent as the IR regulator $m\to 0$ decouples from operators $\mathcal{M}_n$. For $\kappa=0$ and $m=0$, $\mathcal{M}_n$ turn into scaling operators $\mathcal{M}_n^{\rm sc}$ of homogeneity degree $\xi-2$.

The 2-point function equation

$$\mathcal{F}_2(x_1,x_2) \;=\; \mathcal{M}_2^{-1}\mathcal{C}(\tfrac{x_1-x_2}{L}) \tag{31}$$

may be rewritten (with the scaling form d_0^{ij} of d^{ij} and $r\equiv|x_1-x_2|$) as

$$(\mathcal{M}_2\mathcal{F}_2)(r) \;=\; 2r^{1-d}\partial_r\,(Dr^{d-1+\xi}+\kappa r^{d-1})\,\partial_r\,\mathcal{F}_2(r) \;=\; \mathcal{C}(\tfrac{r}{L}) \tag{32}$$

leading to the explicit solution

$$\mathcal{F}_2(r) \;=\; \tfrac{1}{2}\int\limits_r^\infty \frac{\int\limits_0^\rho \mathcal{C}(\frac{\sigma}{L})\,\sigma^{d-1}\,d\sigma}{D\rho^{d-1+\xi}+\kappa\rho^{d-1}}\,d\rho\,. \tag{33}$$

The integration constants were chosen so that $\partial_r\mathcal{F}_2(0)=0$ and $\mathcal{F}_2(\infty)=0$ which is equivalent to the use of the Green function kernel for $\mathcal{M}_2^{-1}$ in eq. (31). Such a choice describes the stationary 2-point function obtained from a localized initial distribution of T by waiting long enough. It is trivial to analyze the integral in eq. (33) explicitly and to see that, first, the $\kappa\to 0$ limit of $\mathcal{F}_2$ exists and, second, that

$$\mathcal{F}_2(r) \underset{\substack{\kappa=0\\ L\ \text{large}}}{=} A_{\mathcal{C}}\,L^{2-\xi} \;-\; \tfrac{\mathcal{C}(0)}{2Dd(2-\xi)}\,r^{2-\xi} \;+\; \mathcal{O}(L^{-2})\,. \tag{34}$$

In particular, for the $2^{\rm nd}$ structure function of T, we obtain

$$S_2(r) \;=\; 2\mathcal{F}(0)-2\mathcal{F}_2(r) \;=\; \tfrac{\mathcal{C}(0)}{Dd(2-\xi)}\,r^{2-\xi} \;+\; \mathcal{O}(L^{-2})$$

in agreement with the naive dimensional analysis of eq. (31). By similar dimensional analysis of eq. (35), we may expect that $S_{2n}(r)\sim r^{n(2-\xi)}$ for large L. This is the Kolmogorov-type prediction and it is incorrect for $n>1$. The first hint of what may go wrong with the dimensional argument may be already seen in eq. (34) where on the right hand side, beside the scaling contribution $-\frac{\mathcal{C}(0)}{2Dd(2-\xi)}\,r^{2-\xi}$, there appears the constant $A_{\mathcal{C}}\,L^{2-\xi}$ diverging when $L\to\infty$. Any divergent contributions to $\mathcal{F}_2$ have to be annihilated by $\mathcal{M}_2$ since the right hand side of eq. (32) is regular when $L\to\infty$ and clearly constants are zero modes of $\mathcal{M}_2$.

Similarly, contributions annihilated by $\mathcal{M}_{2n}$ may appear in the $(2n)$-point correlation function $\mathcal{F}_{2n}$. A more refined analysis shows that, for sufficiently small ξ and for

fixed $x_1,\ldots,x_{2n}$, the $\kappa\to 0$ and $m\to 0$ limits of $\mathcal{F}_{2n}$ exist and are dominated for large L by the contributions of the scaling zero modes of $\mathcal{M}_{2n}^{\rm sc}$:

$$\mathcal{F}_{2n}(\mathbf{x}) \underset{\substack{\nu,m=0\\ L\ \text{large}}}{=} A_{\mathcal{C},2n}\, L^{n(2-\xi)-\zeta_{2n}}\, \mathcal{F}^0_{2n}(\mathbf{x}) + \mathcal{O}(L^{-2+\mathcal{O}(\xi)}) \ + \ \ldots . \tag{35}$$

where $\mathcal{F}^0_{2n}$ is a zero mode of $\mathcal{M}_{2n}^{\rm sc}$ of the homogeneity degree ζ_{2n} given by eq. (27) and the dots denote terms which do not depend on one of the variables x_m and consequently do not contribute to the structure function S_{2n}. The amplitudes $A_{\mathcal{C},2n}$ are non-universal in the sense that they depend on the shape of the forcing covariance $\mathcal{C}$. The zero mode

$$\mathcal{F}^0_{2n}(\mathbf{x}) \ = \ \sum_{\text{permutations}} (x_1-x_2)^2\cdots(x_{2n-1}-x_{2n})^2 \ + \ \mathcal{O}(\xi)\ +\ \ldots .$$

and reduces for $\xi=0$ to a polynomial zero mode of the $(2nd)$-dimensional Laplacian.

The essential tool in arriving at the result (35) is the use of the Mellin transform of eq. (29) with the scaling operator $(\mathcal{M}_{2n}^{\rm sc})^{-1}$, supplemented by the perturbative expansion of the scaling zero modes of $\mathcal{M}_{2n}^{\rm sc}$ in powers of ξ. The analysis has a renormalization group flavor. The perturbative argument is applied to the single scale problem in which the differential operator $\mathcal{M}_{2n}^{\rm sc}$ restricted to scaling functions of a given homogeneity degree is analyzed. Such an operator has a discrete spectrum. The perturbative zero mode information is then plugged into the inverse Mellin transform which assembles different homogeneity degrees. More exactly, the scaling zero modes enter the residues of poles of the Mellin transform of the Green function $(\mathcal{M}_{2n}^{\rm sc})^{-1}$. The above analysis should be contrasted with the direct perturbative expansion of $\mathcal{F}_{2n}$ in powers of ξ which requires perturbative treatment of the Green function of $\mathcal{M}_{2n}^{\rm sc}$, an operator with a continuous spectrum in L^2. The latter expansion is plagued by logarithmic divergences (proportional to powers of $\log L$) which are resummed on the right hand side of (35). Still, the anomalous $\mathcal{O}(\xi)$ contribution to ζ_{2n} may be extracted from the coefficient of $\log L$ in the $\mathcal{O}(\xi)$ term of $\mathcal{F}_{2n}$ proportional to the integral

$$\int \delta_{p,k}\delta_{q,k}\ \frac{\mathrm{e}^{i(p\cdot x+k\cdot y+q\cdot z)/L}}{p^2+k^2+q^2-k\cdot(p+q)}\ \frac{p\cdot q-\frac{(p\cdot k)(q\cdot k)}{k^2}}{|k|^d\,p^2\,q^2}\ \widehat{\mathcal{C}}(p)\,\widehat{\mathcal{C}}(q)\ \text{đ}p\text{đ}k\text{đ}q$$

where $\delta_{p,k}f(p)\equiv f(p+k)-f(p)$. We shall unravel the renormalization group underlying the exponentiation of the logarithmic divergences in the perturbative treatment in powers of ξ of the passive scalar in the next section.

The correlation functions $\mathcal{F}_{2n}$ are not smooth at coinciding points even for $\kappa>0$. Nevertheless

$$\lim_{y\to x}\ (\nabla T)(x)\cdot(\nabla T)(y)\ \equiv\ \epsilon(x)$$

exists inside the correlations for $\kappa>0$ and defines the dissipation field. In particular, the mean dissipation rate $\bar\epsilon\equiv\langle\epsilon(x)\rangle\ =\ \frac{1}{2}\mathcal{C}(0)$, as may be easily seen from eq. (33). $\epsilon(x)$ does not vanish when $\kappa\to 0$ but is given by the dissipative anomaly

$$\lim_{\kappa\to 0}\ \epsilon(x)\ \ =\ \ \lim_{y\to x}\ d^{ij}(x-y)\,(\partial_{x^i}T)(x)\,(\partial_{y^j}T)(y) \tag{36}$$

holding inside correlation functions. The result (35) permits to infer the scaling behavior (26,27) of the structure functions and, together with the dissipative anomaly (36), also the inertial range scaling of the correlations involving the dissipation field. One obtains, for example,

$$\langle\,\epsilon(x)\,\epsilon(y)\,\rangle-\bar\epsilon^2\ \sim\ r^{\zeta_4-2(2-\xi)}\,.$$

LECTURE 4

It has been realized a long time ago [31] that there exists a similarity between the behavior of the 2-point correlation functions of a nearly critical statistical-mechanical systems and of the Fourier transform of the equal-time velocity correlators in a turbulent flow. Both have a scaling regime of power-law decay followed by much stronger decay at infinity. One may then establish the following dictionary [35][11]

critical phenomena	**turbulence**
UV cutoff	integral scale
inverse correlation length	viscous scale
$T - T_c$	viscosity ν
scaling regime	inertial range
anomalous conservation laws	dissipative anomaly

This suggests that, very roughly, the turbulant phenomena look like critical phenomena, provided that we invert the scales interchanging short and long distances or the position and the Fourier spaces. Were this true, the short-distance universality of the critical phenomena (independence of the long-distance behavior of the microscopic details of the system) should be accompanied by the long-distance universality in turbulence (insensitivity of the short-distance behavior to boundary effects or/and details of the energy injection).

The right tool to study the scaling properties of the critical phenomena and to establish their short-distance universality has been provided by the Kadanoff-Wilson renormalization group (RG) [22, 42]. Loosely speaking, the RG idea is to look at the system from further and further away so that its microscopic details are eventually wiped out and many microscopically different models start looking the same. By analogy, it seems [32, 35, 29] that the turbulent systems require an inverse renormalization group (IRG) analysis. By examining them through a stronger and stronger magnifying glass, we would loose their large-scale details from the vision range and should discover a short-distance similarity of different turbulent cascades. The presence of a finite but large correlation length in the nearly critical systems beyond which there is a crossover to the high temperature regime would then correspond to the presence of the short viscous scale in high Reynolds number flows beyond which the dissipative regime sets in.

RG had an enormous success in explaining critical phenomena [41, 39]. Why is it then that IRG never developed beyond the level of a vague idea? Is the analogy between the critical phenomena and turbulence too naive and missing totally the essential points? In the author's opinion the reason is different. RG is not a universal key to all problems as it is sometimes thought. Its effective use requires a correct choice of RG transformations and that, in turn, requires a good understanding of physics of the system. Similarly, an IRG-type analysis of turbulence will require a deep use of knowledge of physics of turbulence. To provide an argument for such a thesis, we shall show that the IRG idea allows to systematize the analysis of Kraichnan's passive scalar described in the previous lecture and opens a possibility to extend it to more complicated systems.

[11] see also [13] for a more recent discussion and more references

Let us start by a short reminder of how one may perform a RG analysis of the long-time long-distance asymptotics of the (nearly) critical dynamics described by the stochastic evolution equation

$$\partial_t T \,=\, D\Delta T \,-\, \tfrac{\delta\mathcal{V}(T)}{\delta T} \,+\, f \tag{37}$$

with $\mathcal{V}(T)=\int V(T(t))\,dt$ and the Gaussian noise f with mean zero and covariance

$$\langle\, f(t,x)\, f(s,y)\,\rangle \,=\, \delta(t-s)\,\mathcal{C}(x-y)\,.$$

The corresponding MSR action is

$$S(R,T) \,=\, S_0(R,T) \,+\, S_1(R,T)$$

where

$$\begin{aligned} S_0(R,T) &= i\!\int\! R(t,x)\,(\partial_t T-\nu\Delta T)(t,x)\,dt\,dx+\tfrac{1}{2}\!\int\! R(t,x)\mathcal{C}(x-y)\,R(t,y)\,dt\,dx\,dy\,,\\ S_1(R,T) &= i\!\int\! R(t,x)\,\tfrac{\delta\mathcal{V}(T)}{\delta T(t,x)}\,dt\,dx\,+\,\tfrac{1}{2}\int \tfrac{\delta^2 V(T(t))}{\delta T(t,x)\delta T(t,x)}\,dt\,dx\,, \end{aligned}$$

see eq. (11). The S_0 part of the action may be used to define the Gaussian "measure"

$$d\mu_{_G}(R,T) \;=\; \mathrm{e}^{-S_0(R,T)}\; DR\,DT\Big/\text{norm.}$$

with covariance G given by the 2-point functions of the forced diffusion (14,15).

Let R,T and $\widetilde{R}$, $\widetilde{T}$ be two copies of Gaussian fields distributed with measure $d\mu_G$. We shall decompose

$$\begin{aligned} R &= \widetilde{R}_{1/\lambda} + \rho\,,\\ T &= \widetilde{T}_{1/\lambda} + \tau \end{aligned} \tag{38}$$

demanding that ρ,τ be Gaussian fields independent of $\widetilde{R},\widetilde{T}$. The rescalings of $\widetilde{R},\widetilde{T}$ are as in eq. (16), i.e. they serve to exhibit the long-time long-distance scaling of the linear forced diffusion. By assumption, we have the factorization

$$d\mu_{_G}(R,T) \,=\, d\mu_{_G}(\widetilde{R},\widetilde{T})\, d\mu_{_{\Gamma_\lambda}}(\rho,\tau)\,. \tag{39}$$

The 2-point functions of ρ,τ building the covariance Γ_λ are the differences of the 2-point functions of R,T and $\widetilde{R}_{1/\lambda}$, $\widetilde{T}_{1/\lambda}$, very much in the spirit of the Pauli-Villars regularization. Explicitly,

$$\langle\,\tau(t_1,x_1)\,\tau(t_2,x_2)\,\rangle \,=\, \int \mathrm{e}^{|t_1-t_2|\,Dk^2\,-\,ik\cdot(x_1-x_2)}\;\tfrac{\widehat{\mathcal{C}}(k)-\widehat{\mathcal{C}}(\lambda k)}{2Dk^2}\,\,{\mathchar'26\mkern-12mu d}k\,. \tag{40}$$

i.e. it is the high-momentum part of the covariance $\langle TT\rangle$. The 2-point functions involving ρ vanish (recall that $\langle RR\rangle=0$ and $\langle TR\rangle$ is scale-invariant) but we shall keep ρ in the formulae which will be later applied in situations with $\rho\neq 0$. The decomposition (38) of T is into the low-momentum part $\widetilde{T}_{1/\lambda}$ and the high-momentum fluctuation τ and allows to define the effective interactions

$$\mathrm{e}^{-S_\lambda(\widetilde{R},\widetilde{T})} \;=\; \int \mathrm{e}^{-S_1(\widetilde{R}_{1/\lambda}+\rho,\,\widetilde{T}_{1/\lambda}+\tau)}\; d\mu_{_{\Gamma_\lambda}}(\rho,\tau) \tag{41}$$

by integrating out the high-momentum fluctuations from the Boltzmann factor e^{-S_1}. The RG transformations $\mathcal{R}_\lambda: S_1\mapsto S_\lambda$ have a semigroup property, $\mathcal{R}_\lambda\circ\mathcal{R}_{\lambda'}=\mathcal{R}_{\lambda\lambda'}$.

We may also integrate out the high-momentum fluctuations in the insertions $F_1(R,T)$ into the MSR functional integral, defining the effective insertions by

$$F_\lambda(\widetilde{R},\widetilde{T}) \;=\; \int F_1(\widetilde{R}_{1/\lambda}+\rho,\,\widetilde{T}_{1/\lambda}+\tau)\;\mathrm{e}^{-S_1(\widetilde{R}_{1/\lambda}+\rho,\,\widetilde{T}_{1/\lambda}+\tau)}\;d\mu_{\Gamma_\lambda}(\rho,\tau)\Big/\,\mathrm{e}^{-S_\lambda(\widetilde{R},\widetilde{T})}. \tag{42}$$

Note that the transformation $F_1\mapsto F_\lambda$ is really the derivative $d\mathcal{R}_\lambda(S_1)$ of the semigroup $\mathcal{R}_\lambda$ and that the RG transformations preserve the expectation values:

$$\langle\, F_1\,\rangle_{S_1} \;=\; \langle\, F_\lambda\,\rangle_{S_\lambda} \tag{43}$$

where $\langle\, F\,\rangle_{S_\lambda} \equiv \int F\,\mathrm{e}^{-S_\lambda}\,/\int \mathrm{e}^{-S_\lambda}$.

The vanishing interaction $S_1=0$ corresponds to a (Gaussian, trivial) fixed point of the semigroup $\mathcal{R}_\lambda$. For non-trivial S_1, the simplest situation occurs if under the action of $\mathcal{R}_\lambda$ it converges to a (possibly trivial) fixed point:

$$S_\lambda \;\underset{\lambda\to\infty}{\longrightarrow}\; S_*\,.$$

The study of the long-distance asymptotics of the correlation functions reduces then to the search of the corresponding scaling fields. Let us explain the latter concept. Let $F(R,T;\mathbf{x})$ be a functional of R, T explicitly dependent on a sequence of space-points $\mathbf{x}$. For example, we may take $F(R,T;\mathbf{x})=\prod\limits_i T(t,x_i)$. Suppose further, that for some exponent ζ_*

$$\lambda^{-\zeta_*}\,(F(\lambda\mathbf{x}))_\lambda \;\underset{\lambda\to\infty}{\longrightarrow}\; F_*(\mathbf{x})\,. \tag{44}$$

Then, tautologically,

$$(F_*(\mathbf{x}))_\lambda \;=\; \lambda^{\zeta_*}\,F_*(\lambda^{-1}\mathbf{x})$$

if in the computation of the effective insertion on the left hand side we use the fixed point interaction S_*. Hence the name: scaling field for $F_*(\mathbf{x})$. Besides, in view of eq. (43),

$$\lim_{\lambda\to\infty}\;\lambda^{-\zeta_*}\,\langle\, F(\lambda\mathbf{x})\,\rangle_{S_1} \;=\; \langle\, F_*(\mathbf{x})\,\rangle_{S_*}$$

giving the long-distance asymptotics of $\langle F(\mathbf{x})\rangle_{S_1}$ (if the fixed-point expectation on the right hand side does not vanish).

The first information about the RG flow $\mathcal{R}_\lambda$ may be obtained by studying its linearization around the Gaussian fixed point. In fact for functionals S_1 polynomially dependent on R,T with local scaling kernels

$$S_\lambda \;=\; d\mathcal{R}_\lambda(0)\,S_1\,+\,\mathcal{O}(S_1^2) \;=\; \lambda^{[S_1]}\,S_1\,+\,\text{lower order polyn.}\,+\,\mathcal{O}(S_1^2) \tag{45}$$

where the (long distance) dimension $[S_1]$ of S_1 is calculated additively with the use of the following table

$$[x]=1\,,\quad [t]=2\,,\quad [T]=1-\tfrac{d}{2}\,,\quad [R]=-1-\tfrac{d}{2}\,.$$

Small irrelevant interactions with $[S_1]<0$ should then die out under the iterated RG transformations[12] resulting in the convergence of S_λ to the trivial fixed point and in the

[12]this may require fine tuning of the lower order terms in S_1

long-time long-distance asymptotics of the correlations as in the linear forced diffusion case. The fate of relevant ($[S_1] > 0$) or marginal ($[S_1] = 0$) interactions cannot be determined by the linear analysis around the trivial fixed point and requires higher order calculations which may show convergence to a non-trivial fixed point situated in a perturbative neighborhood of the trivial one. For example, the nonlinearity with $\mathcal{V}(T) \sim \int T^4$ in eq. (37), describing the Langevin-type dynamics for the ϕ^4 field theory, leads to the interaction $\sim \int RT^3$ of dimension $-1-\frac{d}{2}+3-\frac{3d}{2}+2+d = 4-d$, in agreement with the well known static power counting rendering the nonlinearity irrelevant above 4 dimensions. For $d < 4$, the RG flow is instead governed by a non-trivial fixed point which seems accessible by a perturbative ϵ-expansion in powers of $\epsilon \equiv 4-d$ [42]. An important aspect, essential for the validity of the RG analysis in the above systems, is the approximate locality of the effective interactions S_λ in the position space which means physically that no low-energy interacting modes were removed from the system by integrating out the short-distance fluctuations. Such locality (usually checked only perturbatively) allows to separate the dominant exactly local scaling contributions to S_λ driving the RG dynamics from the remainder strongly damped under the RG flow. Technically, the separation is done by Taylor-expanding the kernels in S_λ in the Fourier space (the approximate locality of the kernels in the position space makes their Fourier transforms smooth).

Suppose that instead of being interested in the long-time, long-distance behavior of the nonlinear forced diffusion, we want to study its short-time, short-distance asymptotics. In particular, we would like to know how the nonlinearity effects the UV asymptotics (19) of the solutions of the linear equation. In order to study this problem, we may set up an (inverse) RG formalism in full analogy with the one described above, with the only difference that we shall use the field rescalings (18) suitable for tracing the short-time, short-distance asymptotics of the linear forced diffusion instead of (16) appropriate for the long-time long-distance behavior. Repeating the decomposition (38) leading to the factorization (39) for the new rescaling (marked in the notation by superscripts rather then subscripts), we shall obtain the fluctuation τ covariance

$$\langle\, \tau(t_1, x_1)\, \tau(t_2, x_2)\,\rangle \;=\; \int \mathrm{e}^{|t_1-t_2|\,Dk^2\,-\,ik\cdot(x_1-x_2)}\;\frac{\widehat{\mathcal{C}}(k)-\lambda^{-d}\widehat{\mathcal{C}}(k/\lambda)}{2Dk^2}\,đk\,. \tag{46}$$

Note that $\widehat{\mathcal{C}}(k) - \lambda^{-d}\widehat{\mathcal{C}}(k/\lambda)$ is the Fourier transform of the $\mathcal{C}(x) - \mathcal{C}(\lambda x)$ so that it corresponds to the long distance part of $\mathcal{C}$. In other words, the decomposition $T = \widetilde{T}^{1/\lambda}+\tau$ is now into the short-distance part and the long-distance fluctuation. Repeating the definitions (41) and (42) for the new scaling, we obtain the IRG semigroup $\mathcal{R}^\lambda$: $S_1 \equiv S^1 \mapsto S^\lambda$ with the derivative $d\mathcal{R}^\lambda(S^1)$: $F_1 \equiv F^1 \mapsto F^\lambda$. If the effective long-distance interactions S^λ converge to a fixed point S^*, the study of the short-distance asymptotics of the correlation functions for the perturbed version of the forced diffusion reduces to the search for the scaling fields

$$\lim_{\lambda\to\infty}\;\lambda^{\zeta^*}\,(F(\mathbf{x}/\lambda))^\lambda \;=\; F^*(\mathbf{x})$$

for which

$$\lim_{\lambda\to\infty}\;\lambda^{\zeta^*}\,\langle\, F(\mathbf{x}/\lambda)\,\rangle_{S^1} \;=\; \langle\, F^*(\mathbf{x})\,\rangle_{S^*}\,.$$

For small interactions S^1,

$$S^\lambda \,=\, d\mathcal{R}^\lambda(0)\,S^1 \,+\, \mathcal{O}((S^1)^2) \,=\, \lambda^{-[S^1]} \,+\, \text{lower order polyn.} \,+\, \mathcal{O}((S^1)^2) \tag{47}$$

where the (short distance) dimension $[S^1]$ is calculated with the use of the new table

$$[x] = 1\,, \quad [t] = 2\,, \quad [T] = 1\,, \quad [R] = -1-d\,.$$

Note the change of the sign in the exponent of λ in eq. (47) as compared to eq. (45)). For example, nonlinearity $\mathcal{V}(T) \sim \int (T)^4$ in eq. (37) leading to the interaction $\sim \int RT^3$ of dimension $-1-d+3+2+d=4$, i.e. irrelevant by power counting. $\int RT^3$ couples, however, to the unstable constant mode of T. Considering instead the gradient-type nonlinearity with $\mathcal{V}(T) \sim \int (\nabla T)^4$ in eq. (37), avoiding coupling to the constant mode, one obtains the interaction $\sim \int R\nabla(\nabla T)^3$ of dimension $-1-d-1+2+d=0$, i.e. marginal in all dimensions. Hence the linearized IRG does not provide any simple hints about the short-time, short-distance asymptotics of the nonlinear forced diffusion. Besides, one should check that the IRG effective interactions S^λ possess in this case a Fourier space locality properties which would allow to separate a finite number of scaling contributions driving the IRG flow. It should be also noticed that the covariance (46), unlike its RG counterpart (40), is not positive which may lead to non-perturbative complications in stabilizing the IRG flow. We shall have nothing more to say about the UV regime of the forced linear diffusion except repeating that its control is an interesting open problem with physical relevance. Below, we shall apply the IRG idea to the passive scalar model of Lecture 3 with milder nonlinearities and milder stability problems.

The mutual relations of RG and IRG lead often to a confusion stemming from the fact that the RG is also used to study the short-distance asymptotics in field theories governed by UV fixed points. Thus there are two contexts in which we apply the standard RG: either we fix the UV cutoff and study the long-distance behavior of the theory by observing stabilization of the system under RG which lowers the momentum cutoff (the statistical-mechanical context) or we start with theories with a larger and larger momentum cutoff and apply RG to lower it to a fixed value, adjusting the parameters of the cutoff theories as to obtain stabilization of the effective theories on the fixed scale (the continuum limit or field theory context). As is well known, the two contexts differ essentially only by a straightforward rescaling and RG used in both of them integrates out the degrees of freedom corresponding to the shortest distances present. Similarly, IRG may be applied to systems forced on long distances in two contexts differing essentially by a simple rescaling. Either we fix the size of the system and the forcing scale (the IR cutoffs) and study the short-distance behavior by trying to exhibit stabilization under IRG which lowers the long-distance cutoffs (the more common situation) or we consider larger and larger systems forced at longer and longer distances and apply IRG to lower the distance cutoff to a fixed value hoping to see stabilization of the effective theories on the fixed distance scale (the infinite-volume limit context). Again, in both contexts the IRG transformations lower the distance cuttoff. In a given situation whether RG or IRG should be used depends on which leads to a finite number of expanding directions in the interaction space[13]. It seems that the limitation of the energy injection to long distances condemns the RG analyses of turbulence employing a modified forcing with a power spectrum and the ϵ-expansion techniques [15, 11, 44].

We would like to study via IRG the short-distance asymptotics of the passive scalar (21) with the random velocity distributed according to eqs. (22-24). Under the rescaling

$$v^\lambda(t,x) \,=\, \lambda^{\xi-1}\, v(t/\lambda^{2-\xi}, x/\lambda)\,, \tag{48}$$

[13]it is *a priori* not excluded that there are exotic systems which yield to both RG and IRG analysis

$$\langle\, v^{\lambda^i}(t,x)\, v^{\lambda^j}(s,y)\,\rangle \;=\; \delta(t-s)\left(2\lambda^\xi D_0\delta^{ij} \;-\; 2d_0^{ij}(x-y) \;+\; \mathcal{O}((\tfrac{m}{\lambda})^{2-\xi})\right)$$

for large λ. Hence $\langle vv\rangle$ exhibits a scaling behavior at short distances, modulo a divergent constant zero mode. One of the problems with the passive scalar is that the term $-v\cdot\nabla T$ couples T to the unstable constant mode of v. A possible solution is to replace eq. (21) by its quasi-Lagrangian version [2]

$$\partial_t T \;=\; -(v-v_0)\cdot\nabla T \,+\, \kappa\Delta T \,+\, f\,. \tag{49}$$

where $v_0(t,x)\equiv v(t,0)$. This changes the stochastic dynamics. The new dynamics may be interpreted as describing the system in the frame moving with one of the material points of the flow [2]. Although eq. (49) corresponds to different dynamical correlations of T, the equal-time correlations do not change [18]. As in Lecture 3, one may obtain for the latter the same equations (29) with $\mathcal{M}_n$ directly in the translation-invariant form (30). Of course, instead of employing the new equation (49), we may keep the original one (21) changing only the velocity covariance to

$$\langle\, v^i(t,x)\, v^j(s,y)\,\rangle \;=\; -2\delta(t-s)\left(d^{ij}(x-y) - d^{ij}(x) - d^{ij}(y)\right)\,. \tag{50}$$

It will be more convenient to choose another $\langle vv\rangle$ covariance, which we shall call mixed,

$$\langle\, v^i(t,x)\, v^j(s,y)\,\rangle \;=\; -\delta(t-s)\left(2d^{ij}(x-y) - d^{ij}(x) - d^{ij}(y)\right) \tag{51}$$

which leads to the equations (29) with $\mathcal{M}_n$ replaced by

$$\begin{aligned}\mathcal{M}'_n \;=\; -\sum_{m=1}^n(\kappa\Delta_{x_m} + d^{ij}(x_m)\,\partial_{x^i_m}\partial_{x^j_m}) \;+\; \sum_{1\le m_1<m_2\le n}&(2d^{ij}(x_{m_1}-x_{m_2}) - d^{ij}(x_{m_1})\\ &-d^{ij}(x_{m_2}))\,\partial_{x^i_{m_1}}\partial_{x^j_{m_2}}\,.\end{aligned}$$

$\mathcal{M}'_n$ again coincide with $\mathcal{M}_n$ in the translational-invariant sector so that the mixed distribution (51) of v leads to the same equal-time correlators of T. The covariance (51) is $\mathcal{O}(\xi)$ for $\kappa=0$ and its use will simplify the perturbative expansion in powers of ξ. Although it lacks positivity, unlike the original covariance or the quasi-Lagrangian one, this will not cause stability problems.

We shall introduce another modification of eq. (21) by replacing it by

$$\partial_t T \;=\; -v\cdot\nabla T \,+\, \kappa\Delta T \,-\, \eta m^2 T \,+\, f \tag{52}$$

where m is the same IR regulator that in eq. (24) and $\eta>0$ is a small fixed constant. The addition of the "mass term" $\sim m^2T$, which changes the long-distance behavior is innocuous for the short-distance one, see below. The expectation values for the passive scalar (52) with the velocity distribution (51) may be generated in the MSR formalism in the following way. Introduce operator

$$M_m \;=\; -d_m^{ij}(x)\,\partial_{x^i}\partial_{x^j} \,+\, \eta m^2\,.$$

M_m is a generator of inhomogeneous superdiffusion (on distance scales $\ll m^{-1}$) and e^{-tM_m} describes the dynamics of the 1-point function of the passive scalar with velocity covariance (51). Let

$$S^0(R,T) \;=\; i\int R(t,x)(\partial_t - M_m)T(t,x)\,dt\,dx \,+\, \tfrac{1}{2}\int R(t,x)\,\mathcal{C}(x-y)\,R(t,y)\,dt\,dx\,dy,$$

$$
\begin{aligned}
S^1(R,T,v) &= i\int R(t,x)\,v(t,x)\cdot\nabla T(t,x)\,dt\,dx\,,\\
S'^1(R,T,v) &= -i\kappa\int R(t,x)\Delta T(t,x)\,dt\,dx\,,\\
S''^1(R,T) &= i\int R(t,x)M_m T(t,x)\,dt\,dx\,.
\end{aligned}
$$

Let $d\mu_{_{D'_m}}(v)$ denote the Gaussian measure with the mixed covariance (51). The equal time $(2n)$-point function $\mathcal{F}_{2n}(\mathbf{x})$ of the scalar T may be represented as the MSR functional integral

$$
\mathcal{F}_{2n}(\mathbf{x}) \;=\; \int\prod_{i=1}^{2n}T(t,x_i)\;\mathrm{e}^{-S(R,T,v)}\,DR\,DT\,d\mu_{_{D'_m}}(v)\;\Big/\;\text{norm.} \tag{53}
$$

where $S\equiv S^0+S^1+S'^1+S''^1$. We have included the $-i\int RM_mT$ term into free action S^0 compensating it by the S''^1 term treated as an interaction. The role of S''^1 is to remove the "tadpole" contractions $\langle(v\cdot\nabla)^2\rangle$ in $\langle(S^1)^2\rangle$. The latter are instead resummed into the free $\langle TR\rangle = i(\partial_t - M_m)^{-1}$ propagators originating from S^0. Reading the field short-distance dimensions from S^0 and, for v, from the short-distance scaling of the $\langle vv\rangle$ covariance, we obtain

$$
[x]=1\,,\quad [t]=2-\xi\,,\quad [T]=1-\tfrac{1}{2}\xi\,,\quad [R]=-1-d+\tfrac{1}{2}\xi\,,\quad [v]=\xi-1\,. \tag{54}
$$

Note that the Kolmogorov value of the velocity exponent $\xi=\frac{4}{3}$ is obtained by equating the dimensions of T and v which is encouraging in view of the fact that the $v\cdot\nabla T$ term of the passive scalar equation is replaced by $v\cdot\nabla v$ in the NS equation. The table (54) gives

$$
[S^1]=0\,,\quad [S''^1]=0\,,\quad [S'^1]=-\xi\,.
$$

We may attempt an IRG analysis of the short-distance behavior of the correlations $\mathcal{F}_{2n}$, basing the IRG transformations on the Gaussian measure $d\mu_{_G}(R,T,v)\sim \mathrm{e}^{-S^0(R,T)}\,DR\,DT\,d\mu_{_{D'}}(v)$ and rescalings corresponding to the table of dimensions (54),

$$
\begin{aligned}
R^\lambda(t,x) &= \lambda^{-1-d+\xi/2}\,R(t/\lambda^{2-\xi},x/\lambda)\,,\\
T^\lambda(t,x) &= \lambda^{1-\xi/2}\,T(t/\lambda^{2-\xi},x/\lambda)
\end{aligned} \tag{55}
$$

and $v^\lambda(t,x)$ as in eq. (48). We could hope that the effective interactions S^λ obtained from the marginal terms $S^1+S''^1$ tend to a fixed point, with the relevant S' term with very small κ destabilizing the IRG trajectory only at very short distances causing eventually a crossover to the dissipative regime. A closer analysis of the effective interactions based on the ξ expansion shows however a lack of convergence of S^λ defined this way to a fixed point [4]. It appears that an infinite number of relevant terms is generated which destabilize the effective trajectory. In such situation, it may seem that IRG fails to predict the short-distance scaling of the scalar correlations in the inertial range. How is it possible then that we still were able to control this scaling for small ξ, as discussed in Lecture 3?

There appears to exist a simple solution to the above paradox. The idea is to exclude the $\frac{1}{2}\int RCR$ term from the free action S^0 expanding in the functional integral (53) $\mathrm{e}^{-\frac{1}{2}\int RCR}=\sum\limits_{n=0}^{\infty}A_n\,F_{_C}^n$ with $A_n\equiv\frac{(-1)^n}{2^n\,n!}$ and $F_{_C}(R)=\int R(t,x)\mathcal{C}(x-y)R(t,y)\,dt\,dx\,dy\,.$
Note that only the $n^{\rm th}$ term of the expansion gives a non-zero contribution to (53)

(the number of the T and R insertions must be equal now). The leftover free action $S'^{0} = i\int R(\partial_t - M_m)T$ leads to the Gaussian measure with the 2-point functions $\langle RR\rangle = 0 = \langle TT\rangle$ and $\langle T(t_1)R(t_2)\rangle = i(\partial_t - Mm)^{-1}(t_1,t_2) = i\theta(t_1-t_2)\,\mathrm{e}^{-(t_1-t_2)M_m}$. It will be more convenient to introduce a cutoff version $d\mu_{_{\Gamma_m^\Lambda}}(R,T)$ of this measure with the 2-point functions

$$\langle\, RR\,\rangle \,=\, 0\,, \qquad \langle\, TT\,\rangle \,=\, 0\,,$$

$$\begin{aligned}\langle\, T(t_1,x_1)\, R(t_2,x_2)\,\rangle \;&=\; i(\partial_t + M_m)^{-1}(t_1,x_1;t_2,x_2) \,-\, i(\partial_t + M_{\Lambda m})^{-1}(t_1,x_1;t_2,x_2)\\ &=\; i\theta(t_1-t_2)\left(\mathrm{e}^{-(t_1-t_2)M_m} - \mathrm{e}^{-(t_1-t_2)M_{\Lambda m}}\right)(x_1,x_2)\,.\end{aligned}$$

For $\Lambda\to\infty$, $\langle T(t_1)R(t_2)\rangle$ tends to[14] $i\theta(t_1-t_2)\,\mathrm{e}^{-(t_1-t_2)M_m}$ except for $t_1=t_2$ for which it always vanishes. Finite Λ introduces a short-time, short-distance cutoff into the heat kernel $\mathrm{e}^{-(t_1-t_2)M_m}$. The functional-integral expression (53) for $\mathcal{F}_{2n}$ may be now rewritten as

$$\begin{aligned}\mathcal{F}_{2n}(\mathbf{x}) \;&=\; A_n \lim_{\Lambda\to\infty} \frac{\int \prod\limits_{i=1}^{2n} T(t,x_i)\; F_{\mathcal{C}}(R)^n\;\mathrm{e}^{-(S^1+S'^1)(R,T,v)}\; d\mu_{_{\Gamma_m^\Lambda}}(R,T)\; d\mu_{_{D'_m}}(v)}{\mathrm{norm.}}\\ &\equiv\; A_n\,\langle \prod_{i=1}^{2n} T(t,x_i)\; F_{\mathcal{C}}(R)^n\,\rangle_{_{S^1+S'^1}}\,. \end{aligned}\qquad (56)$$

The role of the short-distance cutoff Λ is the same as that of the S''^1 term before. It excludes the tadpole contractions $\langle (v(t)\cdot\nabla)^2\rangle$ in $\langle (S^1)^2\rangle$ now forbidden because the cutoff $\langle T(t_1)R(t_2)\rangle$ propagator, unlike its $\Lambda=\infty$ version, vanishes at equal times for all Λ. Note how eq. (56) works for the 2-point function. Expanding on the right hand side $\mathrm{e}^{-S^1+S'^1}$ into the power series, computing the Gaussian expectations and passing to the $\Lambda\to\infty$ limit, one obtains the Neuman series for $\mathcal{M}_2'^{-1}\mathcal{C}$ resulting from treating as a perturbation the second line in the identity

$$\begin{aligned}\mathcal{M}'_2 \;&=\; (M_m)_{x_1} \,+\, (M_m)_{x_2}\\ &+\; (2d^{ij}(x_1-x_2) - d^{ij}(x_1) - d^{ij}(x_2))\,\partial_{x_1^i}\partial_{x_2^j} \,-\, \kappa\Delta_{x_1} \,-\, \kappa\Delta_{x_2}\,.\end{aligned}$$

The higher-point function formula works similarly.

In order to generate the IRG transformations, we shall follow the general rules described before. Introducing the Gaussian fields $\widetilde R, \widetilde T, \widetilde v$ distributed with the measure $d\mu_{_{\Gamma_m^{\Lambda/\lambda}}}(\widetilde R,\widetilde T)\, d\mu_{_{D'_m}}(\widetilde v)$ and decomposing

$$R \,=\, \widetilde R^{1/\lambda} + \rho\,, \qquad T \,=\, \widetilde T^{1/\lambda} + \tau\,, \qquad v \,=\, \widetilde v^{1/\lambda} + w$$

and, for the measures,

$$d\mu_{_{\Gamma_m^\Lambda}}(R,T) \,=\, d\mu_{_{\Gamma_m^{\Lambda/\lambda}}}(\widetilde R,\widetilde T)\;\, d\mu_{_{\Gamma_m^\lambda}}(\rho,\tau)\,, \qquad d\mu_{_{D'_m}}(v) \,=\, d\mu_{_{D'_m}}(\widetilde v)\; d\mu_{_{\delta_m^\lambda}}(w)$$

with

$$\delta_m^\lambda \,=\, D'_m - D'_{\lambda m}\,,$$

[14] the ηm^2 term in M_n plays here the crucial role

we shall define effective interactions $S^\lambda = \mathcal{R}^\lambda(S^1)$ by integrating the long distance fluctuations ρ, τ, w in the Gibbs factor e^{-S^1},

$$\mathrm{e}^{-S^\lambda(\widetilde{R},\widetilde{T},\widetilde{v})} \;=\; \int \mathrm{e}^{-S^1(\widetilde{R}^{1/\lambda}+\rho,\widetilde{T}^{1/\lambda}+\tau,\widetilde{v}^{1/\lambda}+w)}\; d\mu_{\Gamma_m^\lambda}(\rho,\tau)\; d\mu_{\delta_m^\lambda}(w)\,.$$

The effective insertions will be given by the linearization $d\mathcal{R}^\lambda$ of the above IRG flow for interactions, compare eqs. (41,42). Denote $(T(t,x_1)-T(t,x_2))^{2n} \equiv F_{2n}(T;\mathbf{x})$.

Our main claims are as follows:

1. For the molecular diffusivity κ set to zero, the effective interactions S^λ tend to the (non-trivial) limit S^* as $\lambda\to\infty$, at least order by order in the expansion in powers of ξ. At a fixed order, $S^*(\widetilde{R},\widetilde{T},\widetilde{v})$ is given by an explicit integral expressions with kernels approximately local (i.e. fast decaying) in the Fourier space.
2. The limit

$$\lim_{\lambda\to\infty}\; \lambda^{n(2-\xi)}\,(F_{2n}(\mathbf{x}/\lambda))^\lambda \;\equiv\; F^*_{2n}(\mathbf{x})$$

exists and is a scaling field of dimension $n(2-\xi)$ of the fixed-point theory.
3. The limit

$$\lim_{\lambda\to\infty}\; \lambda^{\zeta_{2n}-n(2-\xi)}\,(F^n_{\mathcal{C}})^\lambda \;\equiv\; (F^n_{\mathcal{C}})^*$$

exists and is a scaling field of dimension $\zeta_{2n}-n(2-\xi) = -\frac{2n(n-1)}{d+2}\xi + \mathcal{O}(\xi^2)$, see eq. (27).
4. Finally, the limit

$$\lim_{\lambda\to\infty}\; \lambda^{\zeta_{2n}}\,(F_{2n}(\mathbf{x}/\lambda)\,F^n_{\mathcal{C}})^\lambda \;\equiv\; F^*_{2n,\mathcal{C}}(\mathbf{x})$$

exists and is a scaling field of dimension ζ_{2n}.

The last three results have been established to the first two non-trivial orders in ξ. We shall not discuss here the details of the perturbative analysis referring an interested reader to [4]. Let us only mention the role of the Mellin transform of the kernels entering the effective interactions or insertions in separating the contributions with the lowest dimensions from the reminders strongly damped by the IRG flow.

What are the implication of the above results? First, note that, in view of eq. (56), the relation

$$\lim_{\lambda\to\infty}\; \lambda^{\zeta}_{2n}\,\langle\, F_{2n}(\mathbf{x}/\lambda)\,F_{\mathcal{C}}\,\rangle_{S^1} \;=\; \langle\, F^*_{2n,\mathcal{C}}(\mathbf{x})\,\rangle_{S^*} \qquad (57)$$

implies the anomalous scaling (26): unlike the expectations $\langle F^*_{2n}\rangle_{S^*}$ and $\langle (F^n_{\mathcal{C}})^*\rangle_{S^*}$ which involve unequal numbers of R and T fields, the right hand side of eq. (57) does not vanish. The even more interesting observation is that the anomalous dimensions $\zeta_{2n}-n(2-\xi)$ are carried by the scaling fields $(F^n_{\mathcal{C}})^*$ which are relevant (i.e. of negative dimension) for small ξ and which correspond to the composite operators $F^n_{\mathcal{C}}$. The scaling zero modes of the operators $\mathcal{M}_{2n}$ which played the crucial role in obtaining (26) enter the kernels in the explicit formulae for $(F^n_{\mathcal{C}})^*$. Recall that in statistical mechanics or field theory local composite operators are produced by multiplying fields localized at the same space-point. In the spirit of the scale inversion discussed at the beginning of this lecture, composite fields in turbulence should be obtained by multiplying fields

localized at the same wavenumber in the Fourier space. Since the forcing covariance $\mathcal{C}$ is almost a constant in the inertial range, $F_{\mathcal{C}} \cong \mathcal{C}(0) \int \widehat{R}(t,k)^2|_{k=0}\, dt$ so that the $F_{\mathcal{C}}^n$ insertions are almost local in the wavenumber space. On the other hand, the scaling fields $F_{2n}^*(\mathbf{x})$ corresponding to the insertions $(T(t,x_1)-T(t,x_2))^{2n}$ carry the Kolmogorov (normal) part of the dimension of the structure functions. There are no extra anomalous dimensions appearing in the scaling fields $F_{2n,\mathcal{C}}^*(\mathbf{x})$ which correspond to the products of $F_{2n}(\mathbf{x})$ and $F_{\mathcal{C}}^n$. Their dimensions are just the sum of the dimensions of $F_{2n}^*(\mathbf{x})$ and of the composite scaling fields $(F_{\mathcal{C}}^n)^*$. The IRG picture of the system permits to analyze systematically such operator products in the spirit of operator product expansions of the long distance type (as contrasted with the field-theoretic short-distance OPE's [40] resulting from the RG analysis). Such long-distance OPE's lead to fusion rules of the type studied in [28].

Up to now, we have ignored the relevant S'^1 contribution to the MSR action of the model, proportional to the molecular diffusivity κ. As discussed above, it eventually causes a crossover of the IRG trajectory from the vicinity of the convective fixed point S^* to another one corresponding to the dissipative regime dominating very short distances. It should be interesting to study the crossover and the dissipative regime using IRG. As for the role of the $\eta m^2 T$ term which we have added in the passive scalar equation (52), note that its change gives rise to a term $\sim i\int RT$ in the MSR action of dimension $2-\xi$, i.e. irrelevant at short distances. We could, in fact, analyze directly the $\eta = 0$ case, introducing $\eta > 0$ only in the intermediate IRG steps.

Summarizing. The IRG analysis explains the breakdown of the Kolmogorov scaling of the higher structure functions of Kraichnan's passive scalar for small ξ as due to the appearance of relevant composite scaling fields $(F_{\mathcal{C}}^n)^*$ with a multifractal spectrum of dimensions. These fields do not destabilize the convergence of the effective interactions S^λ to the fixed point S^* since their appearance in S^λ is forbidden by the conservation law imposing the equality of the numbers of R and T fields in interactions (this was not the case in our initial attempt to perform the IRG analysis which kept the $\frac{1}{2}\int R\mathcal{C}R$ term in the action). Note the similarities but also the differences with the RG picture of the critical ϕ^4 theory below 4 dimensions governed by a non-trivial $\mathcal{O}(\epsilon \equiv 4-d)$ fixed point with a modified scaling[15] and a finite number of relevant scaling fields. We believe that although our analysis has used crucially the simplifications inherent in the Kraichnan's model, the above conclusions are robust enough to explain the occurrence of multifractal exponents in other turbulent systems, for example in the passive scalar model with velocities correlated in time [9]. Whether a similar mechanism is responsible for the breakdown of the Kolmogorov scaling in the full-fledged NS turbulence remains to be seen. One of the main problems there is the lack of a small parameter (like ξ above) which will render obtaining reliable numerical values of structure-function exponents very difficult, if not impossible. The situation may be different in weak turbulence [45] which can provide a more gratifying test ground for the IRG ideas.

References

[1] G. K. Batchelor: *Small-Scale Variation of Convected Quantities like Temperature in Turbulent Fluid*. J. Fluid Mech. **5** (1959). 113-133

[15] unlike here where the normal scaling of the 2-point function leads to the absence of wave-function renormalization

[2] V. I. Belinicher, V. S. L'vov: *A Scale Invariant Theory of Fully Developed Hydrodynamic Turbulence*. Sov. Phys. JETP **66** (1987), 303-313

[3] R. Benzi, S. Ciliberto, C Baudet and G. Ruiz Chavaria: *On the Scaling of Three Dimensional Homogeneous and Isotropic Turbulence*. Physica **D 80** (1995), 385-398

[4] D. Bernard, K. Gawędzki, T. Hurd and A. Kupiainen: *Inverse Renormalization Group*. In preparation

[5] D. Bernard, K. Gawędzki and A. Kupiainen: *Anomalous Scaling in the N-Point Functions of a Passive Scalar*. chao-dyn/9601018, Phys. Rev. **E**, in press

[6] L. Caffarelli, R. Kohn and L. Nirenberg: *Partial Regularity of Suitable Weak Solutions of the Navier-Stokes Equations*. Commun. Pure App. Math. **35** (1982), 771-831

[7] M. Chertkov and G. Falkovich: *Anomalous Scaling Exponents of a White-Advected Passive Scalar*. chao-dyn/9509007

[8] M. Chertkov, G. Falkovich, I. Kolokolov and V. Lebedev: *Normal and Anomalous Scaling of the Fourth-Order Correlation Function of a Randomly Advected Scalar*. Phys. Rev. **E 52** (1995), 4924-4941

[9] M. Chertkov, G. Falkovich and V. Lebedev: *Non-Universality of the Scaling Exponents of a Passive Scalar Convected by a Random Flow*. chao-dyn/9601016

[10] S. Corrsin: *On the Spectrum of Isotropic Temperature Fluctuations in an Isotropic Turbulence*. J. Appl. Phys. **22** (1951), 469-473

[11] C. DeDominicis and P.C. Martin: *Energy Spectra of Certain Randomly-Stirred fluids*. Phys. Rev. **A 19** (1979), 419-422

[12] B. Duplantier and A. W. W. Ludwig: *Multifractals, Operator Product Expansion, and Field Theory*. Phys. Rev. Lett. **66** (1991), 247-251

[13] G. Eyink and N. Goldenfeld: *Analogies between Scaling in Turbulence. Field Theory and Critical Phenomena*. Phys. Rev. **E 50** (1994), 4679-4683

[14] A. L. Fairhall, O. Gat, V. L'vov, I. Procaccia: *Anomalous Scaling in a Model of Passive Scalar Advection: Exact Results*. Phys. Rev. **E 53** (1996), 3518-3535

[15] D. Forster, D. R. Nelson and M. J. Stephen: *Large-Distance and Long-Time Properties of a Randomly Stirred Fluid*. Phys. Rev. **A 16** (1977), 732-749

[16] U. Frisch: Turbulence: the legacy of A. N. Kolmogorov. Cambridge University Press, Cambridge 1995

[17] G. Gallavotti: *Some Rigorous Results about 3D Navier-Stokes*. mp_arc/92105

[18] K. Gawędzki, A. Kupiainen: *Universality in Turbulence: an Exactly Soluble Model*. In: Low-dimensional models in statistical physics and quantum field theory. H. Grosse and L. Pittner (eds.), Springer, Berlin 1996, pp. 71-105

[19] K. Gawędzki and A. Kupiainen: *Anomalous scaling of the Passive Scalar*. Phys. Rev. Lett. **75** (1995), 3834-3837

[20] P. Gérard: *Résultats Récents sur les Fluides Parfaits Incompressibles Bidimensionnels*. Asterisque **206** (1992), 411-444

[21] B. I. Halperin and P. C. Hohenberg: *Theory of Dynamical Critical Phenomena*. Rev. Mod. Phys. **49** (1977), 435-479

[22] L. P. Kadanoff: *Scaling Laws for Ising Models near* T_c. Physics **2** (1966), 263-272

[23] A. N. Kolmogorov: *The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds' Numbers*. C. R. Acad. Sci. URSS **30** (1941), 301-305

[24] R. H. Kraichnan: *Small-Scale Structure of a Scalar Field Convected by Turbulence*. Phys. Fluids **11** (1968), 945-963

[25] R. H. Kraichnan: *Anomalous Scaling of a Randomly Advected Passive Scalar*. Phys. Rev. Lett. **72** (1994), 1016-1019

[26] L. Landau and E. Lifshitz: Mécanique des fluides. Mir, Moscow, 1971

[27] J. Leray: *Sur les Mouvements d'un Fluide Visqueux Remplissant l'Espace*. Acta Math. **63** (1934), 193-248

[28] V. L'vov and I. Procaccia: *Fusion Rules in Turbulent Systems with Flux Equilibrium*. Phys. Rev. Lett. **76** (1996), 2898-2901

[29] P. C. Martin and C. DeDominicis: *The Long Distance Behavior of Randomly Stirred Fluids*. Prog. Theor. Phys. Sup. **64** (1978), 108-121

[30] P. C. Martin, E. D. Siggia, H. A. Rose: *Statistical Dynamics of Classical Systems*. Phys. Rev. **A 8** (1973), 423-437

[31] M. Nelkin: *Turbulence, Critical Fluctuations, and Intermittency*. Phys. Rev. **A 9** (1974), 388-395

[32] M. Nelkin: *Scaling Theory of Hydrodynamic Turbulence*. Phys. Rev. **A 11** (1975), 1737-1743

[33] A. M. Obukhov: *Structure of the Temperature Field in a Turbulent Flow*. Izv. Akad. Nauk SSSR, Geogr. Geofiz. **13** (1949), 58-69

[34] L. F. Richardson: Weather prediction by numerical process. Cambridge University Press, Cambridge 1922

[35] H. A. Rose and P. L. Sulem: *Fully Developed Turbulence and Statistical Mechanics*. J. de Phys. **39** (1978), 441-484

[36] V. Scheffer: *An Inviscid Flow with Compact Support in Space-Time*. J. Geom. Analys. **3** (1993), 343-401

[37] A. Shnirelman: *On the Non-Uniqueness of Weak Solution of the Euler Equations*. IHES/M/96/31 preprint

[38] B. Shraiman and E. Siggia: *Anomalous Scaling of a Passive Scalar in Turbulent Flow*. C.R. Acad. Sci. **321** (1995), 279-284

[39] G. Toulouse and P. Pfeuty: Introduction au groupe de renormalisation et à ses applications. Presses Universitaires de Grenoble 1975

[40] K. G. Wilson: *Non-Lagrangian Models of Current Algebra*. Phys. Rev. **179** (1969), 1499-1512

[41] K. G. Wilson: *The Renormalization Group and Critical Phenomena*. Rev. Mod. Phys. **55** (1983), 583-600

[42] K. G. Wilson and J. Kogut: *The Renormalization Group and the ϵ-Expansion*. Phys. Rep. **C 12** (1974), 75-200

[43] W. Wolibner: *Un Théorème sur l'Existence du Mouvement Plan d'un Fluide Parfait, Homogène, Incompressible, Pendant un Temps Infiniment Long*. Math. Z. **37** (1933), 698-726

[44] V. Yakhot and S. A. Orszag: *Renormalization Group Analysis of Turbulence*. J. Sci. Comput. **1** (1986), 3-52 and Phys. Rev. Lett. **57** (1986), 1722-1724

[45] V. E. Zakharov, V. L'vov and G. Falkovich: Kolmogorov spectra in weak turbulence. Springer, Berlin 1992